Spatio-temporal Unstable Chaotic Solutions of the Carleman Kinetic System

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Abstract. The chaotic processes in time and space are investigated explicitly by means of solving the initial-boundary problem for the discrete kinetic equation. The Carleman model is studied. The physical interpretation of this kinetic system is presented. Numerical solutions show series of bifurcations when decreasing the Knudsen number. That leads to the period-doublings and then to the chaotic regimes with the positive senior Lyapunov exponential. The spatial oscillating profiles with the average solutions which differ from the steady profiles are observed. The chaotic character of the oscillations with intermittence is studied. This introduces a basic model for kinetic description of complicated physical processes.

Keywords: Discrete kinetic equations, Carleman model, chaos

INTRODUCTION

Until now the problem of description of unstable flows and turbulence is actual and interesting. The kinetic description introduces new features which could be important especially for compressible flows. The solutions on the basis of the Boltzmann equation are complex (see [1]) and unstable solutions have to be analyzed in details. One of the theoretical ways of investigation such problems is construction of the simple basic model for which one could study the character of instabilities and transition to a complex turbulent chaotic flows. For example the well-known Lorenz model provides the solutions which, strictly speaking, do not describe the real hydrodynamic flows, but contains some characteristics of real turbulece. Note that this model is spatially uniform. The other known model by Kuramoto-Tsuzuki (or Ginzburg-Landau) [2] is originated from the in the chemical reactions and diffusion. This equation is also a simple but in this case it is spatially nonuniform that allows us to reflect main features of real unstable turbulent phenomena. In contrast to the Burgers (see [3]) and Kuramoto-Tsuzuki equations, the Boltzmann kinetic equation contains the linear advective part and the nonlinear (quadratic nonlinearity) right-hand side with the collision integral. Our purpose is to demonstrate chaotic processes in time and space by means of solving the Carleman model when increasing the analog of the Knudsen number. This is the first example of the Discrete Kinetic Equation (DKE) where chaotic regimes have been revealed. The Carleman model possesses some important properties of the kinetic equation. This model equation is a system of two nonlinear equations describing the transfer and the interaction of two types of particles. These equations can be treated as the special kind of the reversible chemical system as well. There are two mathematical incentives to investigate this model. The former is nonintegrability of the system (see [4, 5]) which indicates on the existence of the positive Lyapunov exponentials and the latter is the theorem which states that stationary solutions can be linear unstable. Also, in distinction of the other DKE, this stationary solutions can be easily obtained analytically. The initial-boundary problem for the Carleman model is studied numerically and a series of consecutive bifurcations is obtained with the decreasing the effective Knudsen number. The first bifurcation originates the limit cycle and the following bifurcations destroy it due to the appearance of the new oscillation modes. At last with the increase of the number of oscillations the transition to chaos is observed. The unstable solution including the nonpositive values for the distribution functions. So one can consider this system as a special mathematical model with the chaotic features. Nevertheless these solutions possesses important features of the discrete nonlinear kinetic system and as one can expect for the more number of the discrete velocities (for the large number of the discrete velocities the approximation of the Boltzmann equation takes place) we can obtain the characteristic of the real turbulent flows.

THE FORMULATION OF THE PROBLEM

The Carleman model is a system of two nonlinear equation describing the transfer and interaction of the two types of particles. We will treat these equations as the special kind of the reversible chemical system [6]. Also the investigation of the Carleman equation is very attractive because space non-uniform stationary solutions can be obtained analytically. It has been proved in [7] that stationary solutions of the boundary problem for the Carleman equation can be linear unstable for a certain range of the outer parameters. The Carleman system (see [8]) in the ordinary dimensionless form is as follows

$$\begin{aligned} \frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}&=\frac{1}{\varepsilon}(v^2-u^2),\\ \frac{\partial v}{\partial t}-\frac{\partial v}{\partial x}&=\frac{1}{\varepsilon}(u^2-v^2), \end{aligned} \tag{1}$$

where $\varepsilon \equiv Kn$ is Knudsen number. The boundary problem in dimensionless form is posed for the segment [0,1]:

$$u(0,x)=u_{init}(x),\ \ v(0,x)=v_{init}(x),\ u(t,0)=u_a,\ v(t,1)=v_b. \tag{2}$$

Note that it can be easily proved that the global equilibrium solution of the Carleman system, namely, for $u=v$ is stable at least for u>0, v>0. The steady solution of the mentioned problem has the following form (see [7]):

$$u(x)=A\exp(2cx/\varepsilon)-c/2,\ v(x)=A\exp(2cx/\varepsilon)+c/2.$$

The steady solution depends on two constant A and c. One can see that these value can be uniquely determined through the boundary conditions. Earlier in [7] it has been proved that the steady solution can be unstable. We will consider non-positive solutions of the Carleman system because they can be unstable. This implies that we ought to present the generalized physical interpretation for such unstable solutions. One can try to construct the autocatalytic reversible spatial non-uniform chemical process which will be described by the Carleman system. From the reversibility of the reaction we obtain that one needs to consider at least two components which can transmute each into another. The evolution of the each type of substance concentration is governed by the differential equation. Thus, the Carleman system comprising of the two equations can be adequate candidate for the description of the concentration oscillations in space and time. Consider the narrow cylindrical vessel filled with some liquid solution denoted by P. Due to the fact that the base of the cylinder is much smaller than its height we treat our vessel as one dimensional and use one spatial variable $\tilde{x}$ along the height of the cylinder. The time variable is denoted by $\tilde{t}$. All other dimensional variables will be also denoted with the upper wave line. Suppose that the left border of the cylinder is positively charged and right border is negatively charged. The left border emits the positive ions R^+ which are moving to the right border with the constant velocity $\tilde{\Omega}$. The right border emits the negative ions S^- which are moving to the left border with the same velocity modulus $\tilde{\Omega}$. Consider the reversible chemical reaction in the vessel

$$P+R^+\rightarrow S^-,\qquad P+S^-\rightarrow R^+.$$

Note that liquid solution P becomes the source of the electrons or holes. We do not require the electrical neutrality of the P. Its charge can vary in time due to the chemical reactions and then the conservation law for the total charge of the system consisting of R^+, S^-, P would be satisfied. We deal with the autocatalytic chemical reactions i.e. the velocity of the decay of the each type of substances is proportional to some function of the same type concentrations. Let us denote the concentrations of the number of ions for matters R^+,S^- by the $\tilde{U}(\tilde{t},\tilde{x}),\tilde{V}(\tilde{t},\tilde{x})$ respectively. We consider the autocatalytic process of the second order. Then the velocity of the decay for the concentration R^+ (velocity of the concentration growth for the S^-) is proportional to the $\tilde{\sigma}(\tilde{U}(\tilde{t},\tilde{x})-\tilde{u}_0)^2$, and the velocity of the decay for the concentration S^- (velocity of the concentration growth for the R^+) is proportional to the $\tilde{\sigma}(\tilde{V}(\tilde{t},\tilde{x})-\tilde{v}_0)^2$. Here $\tilde{u}_0$ and $\tilde{v}_0$ are the critical concentrations when the reaction of the transmutation stops. The value $\tilde{\sigma}$ is the cross-section of the reaction. The dynamics of the concentrations is governed by the equations below

$$\begin{aligned} \frac{\partial \tilde{U}}{\partial \tilde{t}}&=-\tilde{\Omega}\frac{\partial \tilde{U}}{\partial \tilde{x}}+\tilde{\sigma}(\tilde{V}-\tilde{v}_0)^2-\tilde{\sigma}(\tilde{U}-\tilde{u}_0)^2,\\ \frac{\partial \tilde{V}}{\partial \tilde{t}}&=+\tilde{\Omega}\frac{\partial \tilde{V}}{\partial \tilde{x}}-\tilde{\sigma}(\tilde{V}-\tilde{v}_0)^2+\tilde{\sigma}(\tilde{U}-\tilde{u}_0)^2. \end{aligned}$$

We make the transformation of the variables by setting $\tilde{u}(\tilde{t},\tilde{x}) = \tilde{U}(\tilde{t},\tilde{x}) - \tilde{u}_0$ and $\tilde{v}(\tilde{t},\tilde{x}) = \tilde{V}(\tilde{t},\tilde{x}) - \tilde{v}_0$. The quantities $\tilde{u}(\tilde{t},\tilde{x})$ and $\tilde{v}(\tilde{t},\tilde{x})$ correspond to the alternation from the critical concentrations and can be either positive or non-positive. Let us denote the coordinates of the vessel's boundary by the $\tilde{x}_1, \tilde{x}_2$ then dimensionless spatial variables are defined as follows: $x = \tilde{x}/\tilde{L}$, where $\tilde{L} = \tilde{x}_2 - \tilde{x}_1$ is the characteristic macroscopic scale and the dimensionless time t equals to $\tilde{t}\tilde{\Omega}/\tilde{L}$. For the dimensionless concentrations we have: $\tilde{n}_0 = |\tilde{u}_a| + |\tilde{v}_b|$, $u = \tilde{u}/\tilde{n}_0, v = \tilde{v}/\tilde{n}_0$, Knudsen number is defined analogously to the ordinary kinetic theory, namely, $\varepsilon = \tilde{\Omega}/(\tilde{\sigma}\tilde{n}_0\tilde{L})$, where the characteristic cross-section $\tilde{\sigma}$ can be assumed as unity and, values $\tilde{u}_a, \tilde{v}_b$ are the boundary conditions. For the Cauchy data and the boundary values we obtain $u_{init} = \tilde{u}_{init}/\tilde{n}_0, v_{init} = \tilde{v}_{init}/\tilde{n}_0, u_a = \tilde{u}_a/\tilde{n}_0, v_b = \tilde{v}_b/\tilde{n}_0$. Then in dimensionless variables we obtain problem (1)-(2).

THE SCENARIO OF THE TRANSITION TO CHAOS IN THE CARLEMAN SYSTEM

We solve the unsteady system (1) numerically and study the unstable solutions with chaotic behavior. For the computations the second order (on time and space) scheme of the predictor-corrector type is used. The conditions of the numerical stability are valid, so we study instabilities originate due to physical processes itself. In the case of stable steady solutions the solutions of the unsteady problem will be in the vicinity of the steady solutions. In the case of unstable steady solutions small errors for the simulation of the system (1) by the finite-difference scheme leads to break out the steady solutions. We use the l_2 measure of the deflections from the stationary solutions:

$$|u|(t_m) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}((u)_i^m - u_{(st)i})^2}, \quad |v|(t_m) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}((v)_i^m - v_{(st)i})^2}.$$

Here the index *i* numerates discrete points in the axis *x*, index *m* numerates discrete points in time, a subscript index *(st)* denotes the steady solution and *N* is the number of spatial points. For the parameter *A*>0 solutions are stable. The case *A*=0 corresponds to spatially uniform solutions and this case is not considered. For the case *A*<0 at some conditions a complex dynamics and the transition to the chaos can be observed. The boundary conditions are uniquely connected with the parameters *A* and *c*. We consider such boundary conditions that *A*=-0.4 and *c*=0.1. For *Kn=0.75* the significant deflections from the steady solutions are observed. The trajectories in phase plane $|u|, |v|$ tend to a point which is the attractor for this case. With decreasing Knudsen number the oscillations about the attractor-point are observed. The amplitude of the oscillations increases when Knudsen number decreases and for *Kn=0.63* the limit cycle appears (see Fig. 1). The radius of this cycle is larger for a smaller Knudsen number. For *Kn=0.57* the first bifurcation of the period-doubling is observed (Fig. 2).

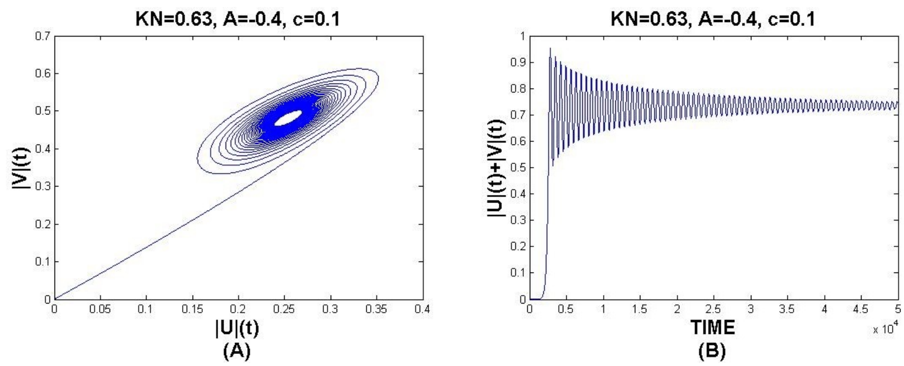


Fig. 1. The origination of the limit cycle.

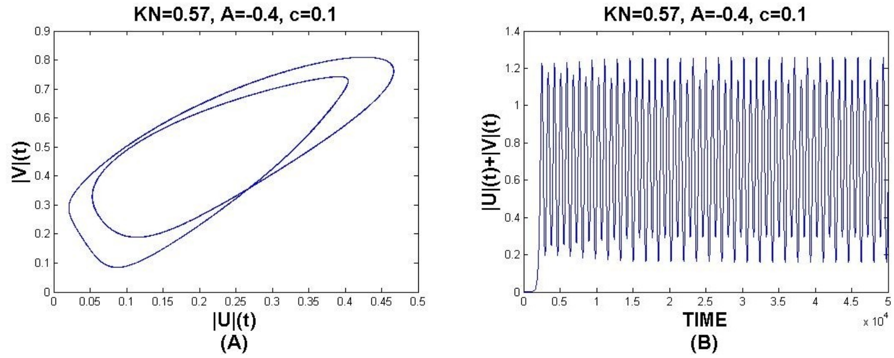


Fig. 2. The first period-doubling bifurcation.

The following bifurcation of the period-doubling is obtained for *Kn=0.562*. The cycle of the period 4 is shown in Fig. 3 for *Kn=0.561*.

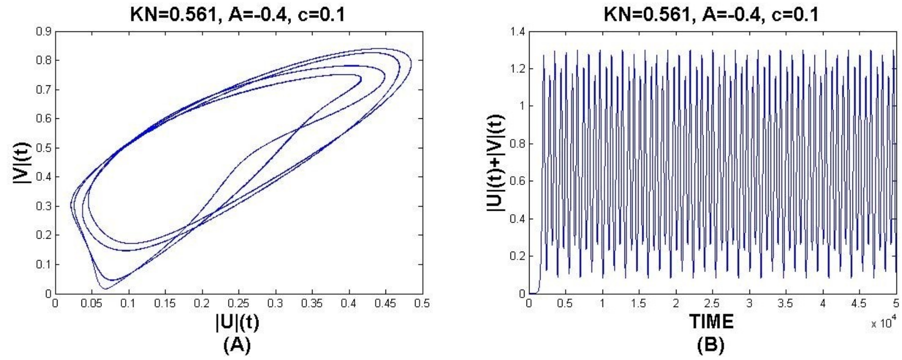


Fig. 3. The second period-doubling bifurcation

For *Kn=0.557* the transition to the chaos is observed (Fig. 4). We made 10 estimations of the senior Lyapunov exponential with aids of the Benettin algorithm [9] for the case showed in Fig. 4 and obtained that it lies in the interval (0.076, 0.086) with mean 0.081, the confidence level is 95%. With the decrease of Knudsen number the situation repeats. Again, between interval of Knudsen numbers (0.546, 0.549) the "window" of periodic regimes and period-doublings are observed and then when Knudsen number approaches to 0.545 the chaos appears. For Knudsen number equals to 0.52 we also estimated the senior Lyapunov exponential, it lies in the interval (0.207, 0.269) with the mean 0.238, the confidence level is 95%. The regular "windows" are observed. The intermittence is shown in Fig. 5a).

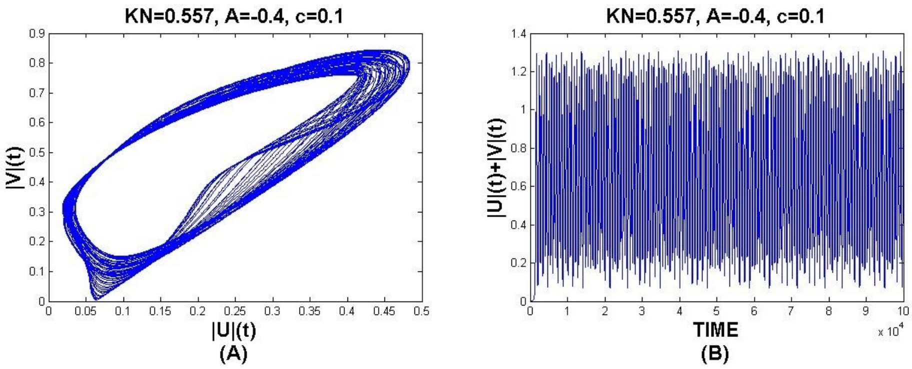
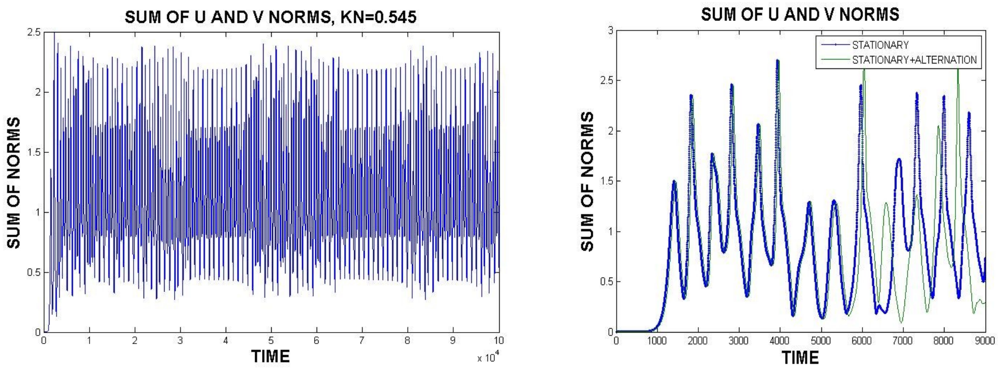


Fig. 4. Chaotic regime.

For confirming the chaotic properties of the solutions we compare two solutions for the different initial conditions Fig. 5b). It is illustrated by the summary characteristic $S_{norm} = |u| + |v|$. In Fig. 5b) the developments of the solutions with the small initial differences (0.01) in the steady solutions are shown. Evolution in time of the sum of l_2 norms u and v at $A=-0.4$, $C=0.1$ for Kn=0.52 for two close initial conditions. One can see that for the initial stage of the process the differences of the solutions are very small but then the differences become large.



a) b)

Fig. 5. Intermittency in time (a) and evolution in time of the sum of l_2 norms u and v at $A=-0.4$, $C=0.1$, Kn=0.52 for two close initial conditions (b).

The special attention is paid to the spatio-temporal behavior of the quantities under consideration. In Fig. 6 the complicated behavior of trajectories in phase space of two values under consideration u and v is observed for different spatial points in the interval [0, 1] and for the Knudsen number Kn=0.52 (A=-0.4, C=0.1). Snapshots for the values (u(t,x),v(t,x)) are depicted in Fig. 7.

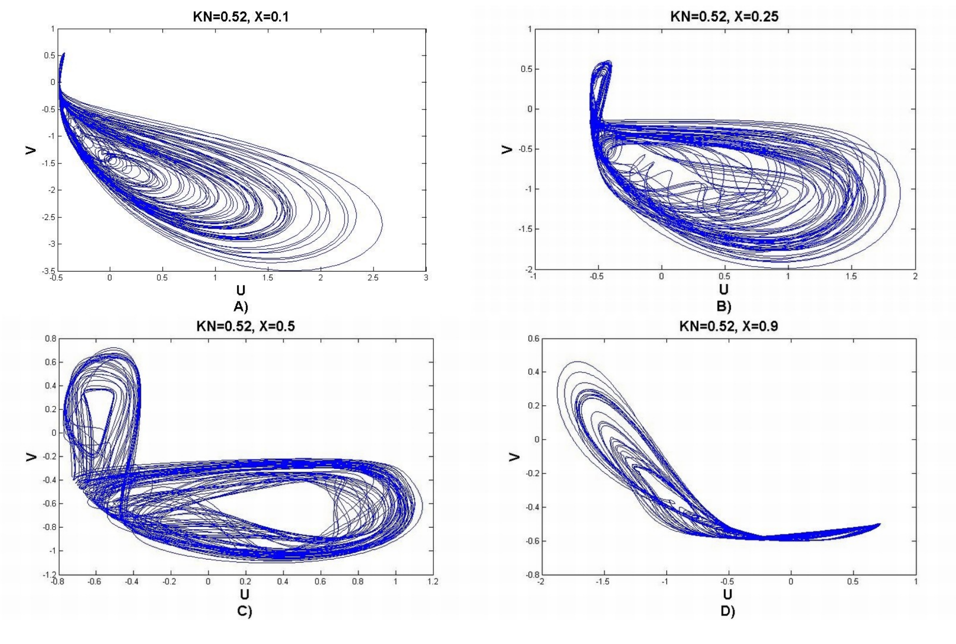


Fig. 6. Behavior of solutions u(t) and v(t) at the fixed values $A= -0.4$, $C=0.1$ for different spatial points.

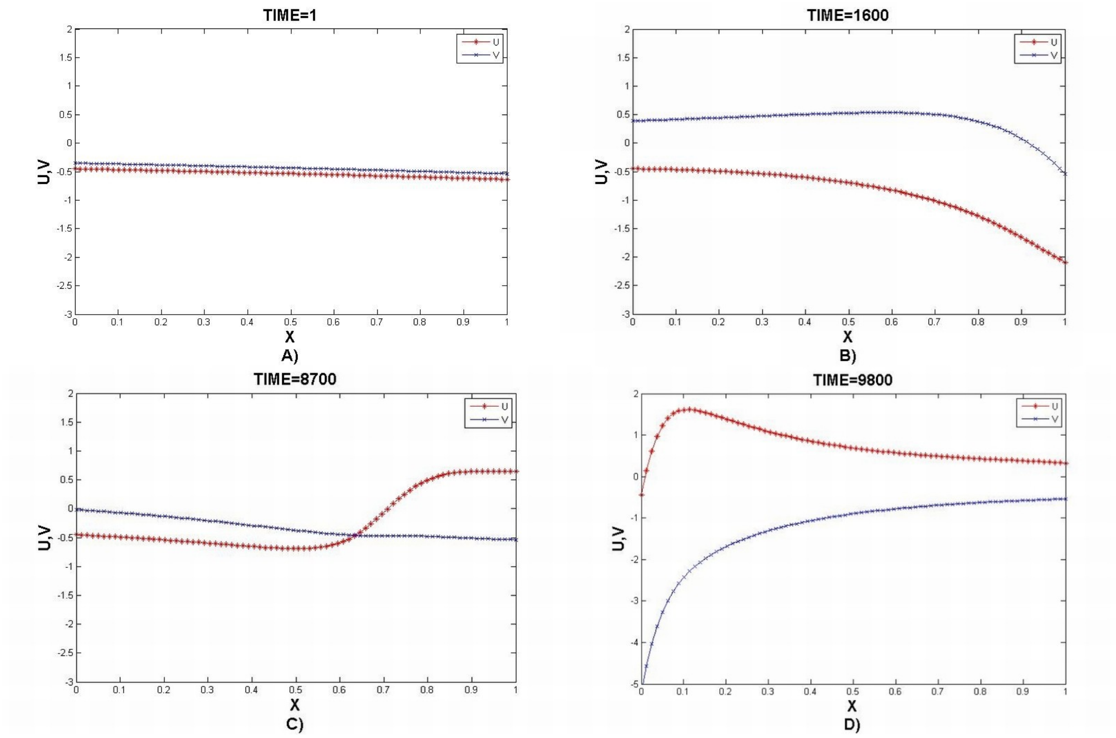


Fig. 7. Spatial profiles of *u* and *v* for different moments of time (presented by the number of the time steps) for the case at *A=-0.4, C=0.1*

CONCLUSION

In the present paper we give the outline of the different stochastic properties that can be obtained for the simplest case of the kinetic equation. We hope that our main result, namely the existence of the cascade of bifurcations leading to the chaos and intermittence, i.e. the Feigenbaum scenario, will be the stimulus for the investigation of the other types of discrete kinetic equations. The future work concerns in particular the increase of the number of the discrete velocities. Really, all DKE are non-integrable and the question is to obtain unstable stationary solutions and then to check chaotic behavior numerically. If the non-regular regimes exist then what scenario transition to the chaos will be observed? At the present moment these questions are opened.

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